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# A Class of Rational Approximations on the Positive Real Axis—A Survey

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Let $f(z)$ be a nonconstant entire function. As usual, write $M(r) = \max_{|z|=r} |f(z)|$; then the order $\rho$ and the lower order $\beta$ of $f(z)$ are [1, p. 8]:

$$\lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\log M(r)}{\log r} = \begin{matrix}\rho\\ \beta\end{matrix} \qquad (0 \leqslant \beta \leqslant \rho \leqslant \infty). \tag{1}$$

If $0 < \rho < \infty$, then the type $\tau$ and the lower type $\omega$ of $f$ are

$$\lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log M(r)}{r^\rho} = \begin{matrix}\tau\\ \omega\end{matrix} \qquad (0 \leqslant \omega \leqslant \tau \leqslant \infty). \tag{2}$$

If $\rho = 0$, then we define the logarithmic order $\rho_l = \Lambda + 1$ of $f$ as

$$\limsup_{r\to\infty} \frac{\log\log M(r)}{\log\log r} = \rho_l = \Lambda + 1 \qquad (0 \leqslant \Lambda \leqslant \infty). \tag{3}$$

If $\rho = 0$, $0 < \Lambda < \infty$, then we define the logarithmic types $\tau_l$ and $\omega_l$ of $f$ as

$$\lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log M(r)}{(\log r)^{\Lambda+1}} = \begin{matrix}\tau_l\\ \omega_l\end{matrix} \qquad (0 \leqslant \omega_l \leqslant \tau_l \leqslant \infty). \tag{4}$$

Let $f(z) \equiv \sum_{k=0}^{\infty} a_k z^k$ be an entire function with nonnegative real $a_k$ $(a_0 > 0)$. Then set

$$\lambda_{0,n} = \lambda_{0,n}\left(\frac{1}{f}\right) = \inf_{\substack{p\in\pi_n \\ p(x)>0,\, 0\leqslant x<\infty}} \left\| \frac{1}{f(x)} - \frac{1}{p(x)} \right\|_{L_\infty[0,\infty)}, \tag{5}$$

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where $\pi_n$ denotes the class of all real algebraic polynomials of degree at most $n$.

Long ago Chebyshev (cf. [20, p. 11]) observed that for every real $g(x)$, continuous on the real axis, for which $\lim_{x\to\infty} g(x) = \lim_{x\to-\infty} g(x)$ (finite), and for $n = 1, 2,\ldots$, there exists a rational function,

$$R_n(x) = \frac{a_0^{(n)} + a_1^{(n)}x + \cdots + a_n^{(n)}x^n}{b_0^{(n)} + b_1^{(n)}x + \cdots + b_n^{(n)}x^n},$$

such that

$$\sup_{-\infty<x<\infty} | g(x) - R_n(x)| \to 0 \qquad (n \to \infty).$$

But Chebyshev never discussed the rate of convergence of $R_n$ to $g$. For such a result see [8a]. Recently much attention has been paid to obtaining upper and lower bounds for the numbers $\lambda_{0,n}(1/f)$, a study initiated by G. Meinardus and R.S. Varga [9] (cf. also [2] and [22].) Applications of these results can be found in Varga's monograph [21]. Our present aim is to present a brief survey of known results and to prove a few new ones.

Section 1 concerns a "converse" theorem (in which a degree of approximation implies smoothness of the approximated function). Section 2 deals mainly with entire functions of positive (finite) order or finite lower order. Entire functions of zero order are discussed in Section 3. Section 4 treats entire functions of infinite order. In the last section we discuss entire functions of the form

$$f(z) = 1 + \sum_{k=1}^{\infty} \frac{z^k}{d_1 d_2 \cdots d_k}, \tag{6}$$

where $d_{k+1} > d_k > 0$, $k = 1, 2,\ldots$ .

## 1

THEOREM 1 [10, THEOREM 3]. *Let $f(x)$ be a real continuous function on $[0, \infty)$, never vanishing there and not a constant there, and assume that there exists a sequence of real polynomials $\{p_n(x)\}_0^\infty$ where each $p_n \in \pi_n$ and never vanishes on $[0, \infty)$ such that*

$$\limsup_{n\to\infty} \left\{\left\| \frac{1}{f(x)} - \frac{1}{p_n(x)} \right\|_{L_\infty[0,\infty)}\right\}^{1/n} < 1. \tag{7}$$

*Then $f$ is the restriction of an entire function $\sum_{k=0}^\infty a_k z^k$ of finite order, so that*

$$\limsup_{r\to\infty} \frac{\log\log M(r)}{\log r} < \infty. \tag{8}$$

*Remarks.* (a) If all $a_k$ are $\geqslant 0$, lim sup in (8) can be replaced by lim (see Theorem 13). Whether this replacement can be made in general is still an open question. (b) If the hypotheses of Theorem 1 hold with (7) replaced by

$$\left\| \frac{1}{f(x)} - \frac{1}{p_n(x)} \right\|_{L_\infty[0,\infty)} \to 0$$

and by the assumption that the coefficients of all $p_n$ are $\geqslant 0$, then $f$ can be shown to be the restriction of an entire function.

## 2

Quite recently A. Schönhage has obtained the following

THEOREM 2 [19]. *For* $f(z) = e^z$,

$$\lim_{n\to\infty} (\lambda_{0,n})^{1/n} = \tfrac{1}{3}. \tag{9}$$

This theorem strengthens a previous result of Cody, Meinardus and Varga ([2], Corollary to Theorem 1, and Theorem 2). It is quite natural to ask how much better one can do in approximating $e^{-x}$ on $[0, \infty)$ using general rational functions than by using reciprocals of polynomials. Recently D. J. Newman has answered this question as follows.

THEOREM 3 [11]. *Let* $P(x)$ *and* $Q(x)$ *be real polynomials of degree* $< n$ $(n \geqslant 1)$, $Q(x) \neq 0$ *throughout* $[0, \infty)$. *Then*

$$\left\| e^{-x} - \frac{P(X)}{Q(x)} \right\|_{L_\infty[0,\infty)} > (1280)^{-n}. \tag{10}$$

In Theorems 4–20 below, every entire function mentioned will be assumed to be of the form $\sum_{k=0}^{\infty} a_k z^k$ $(\not\equiv a_0)$ with $a_k \geqslant 0$, $k = 1, 2,\ldots$, and $a_0 > 0$. (Some results can be stated and proved with $a_0 = 0$ (cf. [9]).)

It is of interest to try to extend (10) to entire functions (Added in proof: Such an extension has just been obtained by A. R. Reddy.) other than $e^{-x}$. Recently Reddy has proved the following related result.

THEOREM 4 [15]. *Let* $f(z)$ *be an entire function of order* $\rho$ $(0 < \rho < \infty)$, *type* $\tau$, *and lower type* $\omega$ $(0 < \omega \leqslant \tau < \infty)$. *Then one cannot find for* $n = 0, 1, 2,\ldots$ *polynomials* $P_n(x)$ *and* $Q_n(x)$ *with nonnegative real coefficients* $(Q_n(0) > 0)$ *and of degree at most* $n$ *for which*

$$\liminf_{n\to\infty} \left\{ \left\| \frac{1}{f(x)} - \frac{P_n(x)}{Q_n(x)} \right\|_{L_\infty[0,\infty)} \right\}^{1/n} < (2\sqrt{2})^{-\tau/(\rho\omega)}. \tag{11}$$

*Remark.* Theorems 2 and 4 imply that in approximating $e^{-x}$ under the uniform norm on $[0, \infty)$ by reciprocals of polynomials of degree $\leqslant n$ we can achieve a smaller error (for large $n$) than in approximating by general rational functions (with degrees of numerator and denominator $\leqslant n$) having nonnegative real coefficients.

Modifying the hypotheses of Theorem 4, we have the following:

THEOREM 5 [17]. *Let $f(z)$ be an entire function of order $\rho$ and maximal type or of (finite) order $> \rho$. If $P_n(x)$ and $Q_n(x)$ are, for $n = 0, 1, 2,...$, polynomials with nonnegative real coefficients ($Q_n(0) > 0$) of degree at most $n$, then*

$$\limsup_{n\to\infty} \left\{ \left\| \frac{1}{f(x)} - \frac{P_n(x)}{Q_n(x)} \right\|_{L_\infty[0,\infty)} \right\}^{\rho/n} \geqslant \frac{1}{2.75}. \tag{12}$$

THEOREM 6. *Let $f(z)$ be an entire function of either infinite order or of finite order but not of regular growth. If $(P_n(x))_{n=0}^{\infty}$ and $(Q_n(x))_{n=0}^{\infty}$ are as in Theorem 5, then*

$$\limsup_{n\to\infty} \left\{ \left\| \frac{1}{f(x)} - \frac{P_n(x)}{Q_n(x)} \right\|_{L_\infty[0,\infty)} \right\}^{1/n} \geqslant 1. \tag{13}$$

The proof of the last theorem is very similar to that of Theorem 2 of [4].

Returning to approximation by reciprocals of polynomials, we state the following result [9, Theorems 2 and 3].

THEOREM 7. *Let $f(z)$ be an entire function of order $\rho$ $(0 < \rho < \infty)$ satisfying the further assumption that*

$$0 < \lim_{r\to\infty} \frac{\log M(r)}{r^\rho} = \tau < \infty.$$

*Then*

$$\frac{1}{4 \cdot 2^{1/\rho}} \leqslant \limsup_{n\to\infty} (\lambda_{0,n})^{1/n} \leqslant \frac{1}{2^{1/\rho}}. \tag{14}$$

Recently Reddy [13, 14 and 17a] has succeeded in generalizing (14) as follows:

THEOREM 8. *Let $f(z)$ be an entire function of order $\rho$ $(0 < \rho < \infty)$, type $\tau$, and lower type $\omega$. Then*

(i) $\limsup_{n\to\infty} [\log n/\log(\lambda_{0,n}^{-1})] \leqslant \rho$.

(ii) *If $\tau < 2\omega$, then*

$$\limsup_{n\to\infty} (\lambda_{0,n})^{1/n} \leqslant \left(\frac{\tau}{2\omega}\right)^{x_2/(\rho x_1)}, \tag{15}$$

*where $x_1$ is the largest and $x_2$ the smallest root of the equation*

$$x \log(x/e) + (\omega/\tau) = 0. \tag{16}$$

(iii) *If* $0 < \omega \leqslant \tau < \infty$, *then*

$$\liminf_{n\to\infty} (\lambda_{0,n})^{1/n} \geqslant [4(2\tau/\omega)^{1/\rho} - 1]^{-2}. \tag{17}$$

It is interesting to know whether one can extend Theorems 7 and 8 to wider classes of entire functions. In this connection Meinardus, Reddy, Taylor and Varga [10, Theorem 6] have obtained the following theorem.

THEOREM 9. *Let $f(z)$ be an entire function of order $\rho$ $(0 < \rho < \infty)$, type $\tau$, and lower type $\omega$ $(0 < \omega \leqslant \tau < \infty)$. Then*

$$\limsup_{n\to\infty} (\lambda_{0,n})^{1/n} < 1. \tag{18}$$

(For a simpler proof see [17a].)

Quite recently, by adopting slightly different techniques, Erdös and Reddy [6, Theorem 3] have obtained the following theorem.

THEOREM 10. *Let $f(z)$ be an entire function of order $\rho$ $(0 < \rho < \infty)$, type $\tau$, and lower type $\omega$ $(0 < \omega \leqslant \tau < \infty)$. Then*

$$0 < \left(\frac{e\omega^2}{e^{\omega/[\tau(e+1)]}\tau^2(e+1)\,4^\rho}\right)^{x_1/(\rho x_2)} \leqslant \liminf_{n\to\infty} (\lambda_{0,n})^{1/n}, \tag{19}$$

$$\limsup_{n\to\infty} (\lambda_{0,n})^{1/n} \leqslant \exp\left(\frac{-\omega}{(e+1)\,\rho\tau}\right) < 1, \tag{20}$$

*where $x_1$ and $x_2$ are as in Theorem* 8 (*cf. Theorem* 14).

*Remarks.* (a) There exist entire functions which fail to satisfy the assumptions of Theorem 10 but for which we still have (cf. [6])

$$0 < \liminf_{n\to\infty} (\lambda_{0,n})^{1/n} \leqslant \limsup_{n\to\infty} (\lambda_{0,n})^{1/n} < 1. \tag{21}$$

(b) It is quite natural to ask whether one can get any results involving the lower order $\beta$ of an entire function without making any restriction on the order. In this connection Erdös and Reddy [5, Theorem 2; 7a, Theorem 2] have proved the following theorem.

THEOREM 11. *Let $f(z)$ be an entire function of finite lower order $\beta$. Then*

$$\liminf_{n\to\infty} (\lambda_{0,n})^{1/n} \leqslant e^{[-\beta(e+1)]^{-1}} \;(= 0 \quad \text{if} \quad \beta = 0). \tag{22}$$

*If the order $\rho$ of $f$ satisfies $\beta < \rho < \infty$, then* $\liminf_{n\to\infty} (\lambda_0, n)^{1/n} = 0$.

We mention here the following result of Erdös and Reddy [5, Theorem 1].

THEOREM 12. *Given a real sequence $(g(n))_{n=0}^{\infty}$ with $g(n) \to \infty$, there is an entire function of infinite order for which, for infinitely many $n$,*

$$\lambda_{0,n} \leqslant 1/g(n). \tag{23}$$

THEOREM 13. *Let $f(z)$ be an entire function of lower order $\beta$ and order $\rho$ $(\beta < \rho)$.*[1] *Then*

$$\limsup_{n\to\infty} (\lambda_{0,n})^{1/n} \geqslant 1. \tag{24}$$

*Proof.* Since $f(z)$ is of irregular growth, for each $s > 0$ there exists an arbitrarily large real $t_s$ for which

$$f(t_s(1 + s^{-1})) \geqslant [f(t_s)]^s. \tag{25}$$

(25) implies the conclusion (cf. [4, Theorem 2]).

THEOREM 14. *Let $f(z)$ be an entire function and suppose there exist constants $\delta > 1$, $c > 1$, $\epsilon > 0$ and $0 < c_1 < c_2 < 1$, for which, for all large $r$,*

$$M(r\delta) \geqslant \{M(r)\}^{\theta}, \qquad \textit{where} \quad \theta = \frac{c_2}{c_1} + \frac{\log(4\delta - 2)}{c_1 \log c} + \epsilon. \tag{26}$$

*Then for every sequence $(p_n(x))_{n=0}^{\infty}$, where each $p_n(x)$ is a real polynomial of degree $\leqslant n$, positive throughout* $[0, \infty)$, *we have*

$$\liminf_{n\to\infty} \left\{ \left\| \frac{1}{f(x)} - \frac{1}{p_n(x)} \right\|_{L_\infty[0,\infty)} \right\}^{1/n} \geqslant c^{-\theta} > 0. \tag{27}$$

*Proof.* Suppose for some such sequence $(p_n(x))_{n=0}^{\infty}$, (27) were false. Then for $n = n_1, n_2, \ldots$, where $0 < n_1 < n_2 < \ldots$, we would have

$$\left\| \frac{1}{f(x)} - \frac{1}{p_n(x)} \right\|_{L_\infty[0,\infty)} < c^{-n\theta}. \tag{28}$$

For every $n_q$ sufficiently large, there is an $r_{n_q} > 0$ such that

$$f(r_{n_q}) = c^{c_1 n_q}, \tag{29}$$

[1] Observe that under the hypotheses of Theorem 10, we have, unlike here, $\beta = \rho$.

and furthermore,

$$p_{n_q}(r_{n_q}) < c^{c_2 n_q}, \tag{30}$$

for otherwise (28) would be contradicted. Now using (26) and (29), we obtain

$$f(r_{n_q}\delta) \geqslant c^{n_q \theta c_1}. \tag{31}$$

But for $x = r_{n_q}\delta$, by using a result of Remez (cf. [8, 534–535]), we get

$$p_{n_q}(x) < (4\delta - 2)^{n_q} (c^{c_2 n_q}). \tag{32}$$

From (31) and (32) we have for that $x$,

$$c^{-n_q\theta} < (4\delta - 2)^{-n_q} (c^{-n_q c_2}) - c^{-n_q \theta c_1} < \frac{1}{p_{n_q}(x)} - \frac{1}{f(x)}, \tag{33}$$

contradicting (28).

## 3

For functions of order 0, Meinardus, Reddy, Taylor and Varga [10, Theorem 7] have obtained the following result.

THEOREM 15. *Let $f(z)$ be an entire function of logarithmic order $\rho_l = \Lambda + 1$ $(0 < \Lambda < \infty)$, and logarithmic types $\tau_l$ and $\omega_l$ $(0 < \omega_l \leqslant \tau_l < \infty)$. Then*

$$\lim_{n\to\infty} (\lambda_{0,n})^{1/n} = 0. \tag{34}$$

Recently Reddy (cf. [12, 16]) has improved (34) as follows.

THEOREM 16. *Let $f(z)$ satisfy the assumptions of Theorem 15. Then*

$$\exp\left(\frac{-\Lambda}{(\Lambda + 1)[(\Lambda + 1)\,\tau_l]^{1/\Lambda}}\right) \leqslant \limsup_{n\to\infty} (\lambda_{0,n})^{n^{-[1+\Lambda^{-1}]}} < 1. \tag{35}$$

The following result is valid for a wider class of entire functions.

THEOREM 17 [6, Theorem 6]. *Let $f(z)$ be an entire function for which*

$$1 \leqslant \limsup_{r\to\infty} \frac{\log\log M(r)}{\log\log r} = \Lambda + 1 < \infty. \tag{36}$$

*Then for every* $\epsilon > 0$,

$$\liminf_{n\to\infty} (\lambda_{0,n})^{n^{-[1+(A+\epsilon)^{-1}]}} < 1. \tag{37}$$

*Remark.* If (36) holds with lim replacing lim sup, then we can replace lim inf in (37) by lim sup. The proof of this is very similar to that of Theorem 7* in [12].

## 4

Most of the above results concerned entire functions of finite order.

Recently, Erdös and Reddy [3–7] have developed a method by which one can treat any entire function $\sum_{k=0}^{\infty} a_k z^k$ for which all $a_k \geqslant 0$, $k = 1, 2,\ldots$, $a_0 > 0$. For example, they have shown [6, Theorem 1] the following:

THEOREM 18. *Let $f(z)$ be an entire function. Given $\epsilon > 0$ and a positive integer $j$, there exist infinitely many $n$ for which*

$$\lambda_{0,n} \leqslant \exp\left(\frac{-n}{(l_1 n)(l_2 n) \cdots (l_j n)^{1+\epsilon}}\right), \tag{38}$$

*where*

$$l_j n = \log\log \cdots \log n \ (j \text{ times}).$$

On the other hand, Theorem 1 implies

THEOREM 19. *Let $f(z)$ be an entire function of infinite order. Then*

$$\limsup_{n\to\infty} (\lambda_{0,n})^{1/n} \geqslant 1.$$

(For a direct proof, see [4]).

## 5

THEOREM 20. *Let $f(z) = 1 + \sum_{k=1}^{\infty} z^k/(d_1 d_2 \cdots d_k)$, with $d_{k+1} > d_k > 0$, $k = 1, 2,\ldots$, be an entire function of finite order $\rho$. Then for any $\epsilon > 0$, we have for all large $n$,*

$$\frac{d_1 d_2 \cdots d_n}{2^{4n} d_n^{2(\rho+\epsilon)} d_{n+1} d_{n+2} \cdots d_{2n}} \leqslant \lambda_{0,2n-1} \leqslant \frac{d_1 d_2 \cdots d_n}{d_{n+1} d_{n+2} \cdots d_{2n}} \left(\frac{d_{2n+1}}{d_{2n+1} - d_{2n}}\right). \tag{39}$$

This result is due to Erdös and Reddy [6, Theorem 4].

Theorem 20 applies to many functions which fail to satisfy the assumptions of some earlier theorems. For example [6], let

$$f(z) = 1 + \sum_{k=1}^{\infty} \frac{z^k}{2^{\log 2}\, 3^{\log 3} \cdots k^{\log k}}.$$

For this function, $\Lambda = \infty$, hence Theorems 15–17 are inapplicable. But (39) readily implies

$$\lim_{n\to\infty} (\lambda_{0,n})^{1/(n\log n)} = \tfrac{1}{4}.$$

Let $f(z) = 1 + \sum_{k=1}^{\infty} z^k/(2^2 3^3 \cdots k^k)$. For this function, $\Lambda = 1$, $\tau_l = 0$; hence Theorems 15 and 16 are not applicable. But (39) implies

$$\lim_{n\to\infty} (\lambda_{0,n})^{1/(n^2\log n)} = e^{-1/4}.$$

Let $f(z) = \sum_{k=0}^{\infty} z^k/\delta^{2^k}$ $(1 < \delta < \infty)$. For this function $\Lambda = 0$. One gets from (39),

$$\lim_{n\to\infty} (\lambda_{0,n})^{1/2^{n+1}} = 1/\delta.$$

## Concluding Remarks

(a) An interesting question is to what extent the requirement $a_k \geqslant 0$, $k = 1, 2,\ldots$, $a_0 > 0$ can be removed or modified in the above theorems. See the last sentence in the paragraph following Theorem 3, as well as [7a and 18].

(b) For extensions to nonentire functions, and for some results on series with gaps, (cf. [7]).

(c) For Müntz-type results, cf. [7a].

(d) Approximation to reciprocals of some entire functions by reciprocals of exponential polynomials, as well as by reciprocals of linear combinations of certain entire functions of small growth has been studied in [17b].

(e) One can show that if $0 < \beta \leqslant \rho < \infty$, then

$$\beta/\rho \liminf_{n\to\infty} [\log\log(\lambda_{0,n}^{-1})]/\log n \leqslant \limsup_{n\to\infty} [\log\log(\lambda_{0,n}^{-1})]/\log n \leqslant \rho/\beta.$$

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